ICS 6B
Boolean Algebra & Logic

Lecture Notes for Summer Quarter, 2008

Michele Rousseau

Final Review

(Some slides inspired and adapted from Alessandra Pantano)

Chapter 12

- How do you get a derivation from a Phrase Structured Grammar (G=V,T,S,P)

Suppose a phrase-structured grammar has productions:

S →ABa, (1) A →BB, (2) B →ab, (3) AB →b (4)

Find the derivation of BBaba:

S →(1) ABa →(4) ba

ABa →(2) BBBa →(3), (3), (3) abababa

ABa →(3) Aaba →(2) BBaba →(3), (3)

Answer: ABa ⇒ Aaba ⇒ Bbaba

or ABa ⇒ BBBa ⇒ BBaba

Construct the Language of a Grammar

G ={V, T, S, P} with

- V={S, A, a, b}
- T={a, b}
- S= the start element
- P={S⟶aA, S ⟶b, A=aa}

Find L(G)

S⟶ aA ⟶ aaa

S ⟶ b

L(G) = {b, aaa}

Different types of Grammars

Type	AKA	Productions Allowed
0		Any production – no restrictions
1	Context – sensitive	Change 1 (non-terminal symbol) between 2 strings AND S⟶λ lAr⟶lwr A is a N (non-terminal) symbol lr is a string of T & N symbols (may be λ) w is a string of T & N symbols (can't be empty)
2	Context-free	Change **1** non-terminal symbol A⟶lr A is a single N terminal symbol lr is a string of T & N symbols (may be λ)
3	regular	Change **1** non-terminal symbol → very restricted how it can be changed A⟶aB; A⟶a & S⟶λ A & B are non terminal, a is terminal

Example

- V={S,A,B,a,b}
- T={a,b}
- P=
  - (1)S⟶Aba, AB ⟶ a,
  - (2)S ⟶λ, aAb ⟶aab
  - (3)S⟶λ, A⟶aa
  - (4)S⟶λ, A ⟶b

Type	AKA	Productions Allowed
0		no restrictions
1	Context – sensitive	S⟶λ lAr⟶lwr
2	Context-free	A⟶lr
3	regular	A⟶aB; A⟶a & S⟶λ

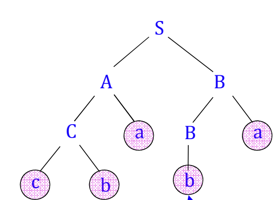
- **G1) type 0, but not type 1**
  - AB is 2 N symbols
- **G2) type 1, but not type 2**
  - Not a single N symbol (has some T symbols too)
- **G3) type 2, but not type 3**
  - Goes to two T & N strings

G4) type 3

S ⟶ AB ⟶CaBa ⟶cbaba

- Given P:

  S⟶AB
  1. A ⟶Ca
  2. B ⟶Ba
  3. B ⟶Cb
  4. B ⟶b
  5. C ⟶cb
  6. C ⟶b



- **Construct a derivation tree for cbaba**

We see that cbaba can be derived from:

S ⇒ AB ⇒ CaBa ⇒ cba

Terminal elements are at the end of each branch so we are done

Vending Maching I/O and States

Possible inputs	Possible outputs
Nickel (5)	nothing (N)
O-button (O)	Nickel (5)
A-button (A)	OJ
	A

Possible states

- initial state s_0 (no money in the machine)
- state s_1 (5c in the machine)
- state s_2 (10c in the machine)
- state s_3(15 in the machine)

Remember Vending Machine

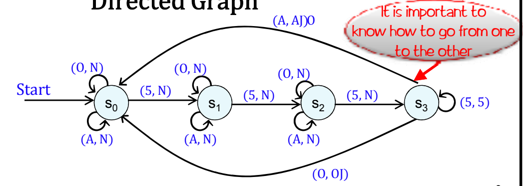
Inputs
(5)Nickel
(O) O-btn
(A)-A-Btn

Outputs
(N) Nada
(5) Nickel
(OJ)
(AJ)

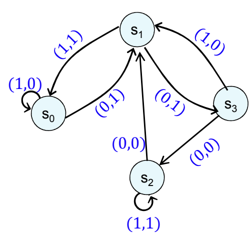
Table

	States	Transition – f(s,i) 5	O	A	Output – g(s,i) 5	O	A
0	s_0	s_1	s_0	s_0	N	N	N
5c	s_1	s_2	s_1	s_1	N	N	N
10c	s_2	s_3	s_2	s_2	N	N	N
15c	s_3	s_3	s_0	s_0	5	OJ	AJ

Directed Graph



Going from Graph to Table



States	Transition– f(s,i) 0	1	Output – g(s,i) 0	1
s_0	s_1	s_0	1	0
s_1	s_3	s_0	1	1
s_2	s_1	s_2	0	1
s_3	s_2	s_1	0	0

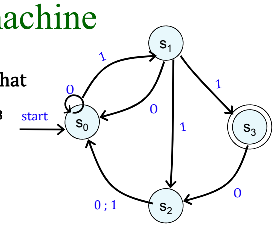
- Also, be able to determine output

Example
What is the output of 1110?
0001

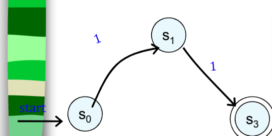
Also be able to do this with automaton

What strings are recognized by the following machine

- Only 1 final state s_3
- Think of all possible paths that originate at s_0 and end at s_3



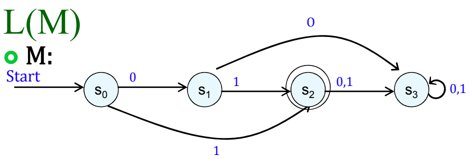
- We can loop as many times as we want, and then we must follow



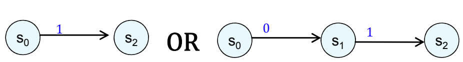
This says the strings recognized by the machine are all of the for 0^n 11, with $n \geq 0$

Given a automaton determine L(M)

- M:



- There's only 1 final state (s_2). So we look for paths from s_0 to s

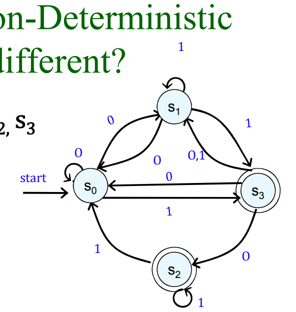


- => L(M) ={0,01}

How is a Non-Deterministic automaton different?

- Final states: s_2, s_3

States	Inputs 0	1
s_0	s_0, s_2	s_3
s_1	s_0	s_1, s_3
s_2		s_0, s_2
s_3	s_0, s_2, s_2	s_1



Notice that the transition function with state s_2 and input 0 produces the empty set of states: f(s_2 , 0) does not give any next state

Chapter 1

- Determine whether this proposition is a tautology: $((p \rightarrow \neg q) \wedge q) \rightarrow \neg p$

p	q	$\neg p$	$\neg q$	$p \rightarrow \neg q$	$\wedge\, q$	$\rightarrow \neg p$
T	T	F	F	F	F	T
T	F	F	T	T	F	T
F	T	T	F	T	T	T
F	F	T	T	T	F	T

- Also know how to prove equivalences and how to show using identities => know the identities

Chapter 1:

- In the questions below write the statement in the form "If ..., then ...."

"The team wins if the quarterback can pass."

If the quarterback can pass, then the team wins.

Write the contrapositive, converse, and inverse of the following: "You sleep late if it is Saturday"

Contrapositive: $\neg q \rightarrow \neg p$

If you do not sleep late, then it is not Saturday.

Converse: $q \rightarrow p$

If you sleep late, then it is Saturday.

Inverse: $\neg p \rightarrow \neg q$

If it is not Saturday, then you do not sleep late.

English to Logical Expressions

F(x): x is a fleegle
S(x): x is a snurd
T(x): x is a thingamabob
U={fleegles, snurds, thingamabobs}

" Everything is a fleegle"

$\forall x\, F(x) \equiv \neg \exists x \neg F(x)$

" Nothing is snurd"

$\forall x \neg S(x) \equiv \neg \exists x\, S(x)$

" All fleegles are snurds"

$\forall x [F(x) \rightarrow S(x)]$

$\equiv \forall x [\neg F(x) \vee S(x)]$

$\equiv \forall x \neg [F(x) \wedge \neg S(x)]$

$\equiv \neg \exists x [F(x) \wedge \neg S(x)]$

CHAPTER 2

Remember the Cartesian Product?

Let A={1,2} , B= {a,b,c}, C= {y,z}

What is A x B x C

- A x B x C= {(1,a,y), (1,a,z), (1,b,y), (1,b,z), (1,c,y), (1,c,z), (2,a,y), (2,a,z), (2,b,y), (2,b,z), (2,c,y), (2,c,z)}

Thus A x B x C is all possible ordered tuples (a,b,c) where $a \in A$, $b \in B$, and $c \in C$

Operations on Sets

U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

A= {1, 2, 3, 4, 5}

B = {4, 5, 6, 7, 8}

A ∪ B= {1, 2, 3, 4, 5, 6, 7, 8}

A ∩B = {4, 5}

A^c = {0, 6, 7, 8, 9, 10}

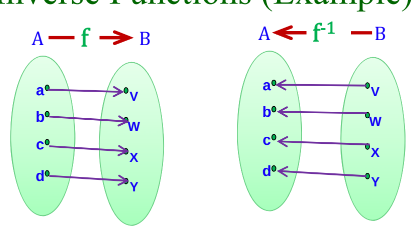
B^c = {0, 1, 2, 3, 9, 10}

A - B = {1, 2, 3}

B - A = {6, 7, 8}

A⊕B = {1, 2, 3, 6, 7, 8}

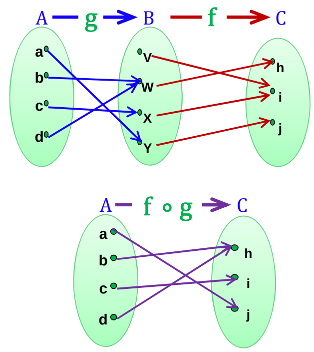
Inverse Functions (Example)



A bijection is called *invertible* because you can define the inverse function. To be invertible it must be a bijection.

Composition Example

If $f(x)=x^2$ and
$g(x)=2x+1$
Then
$f(g(x))=f(2x+1)$
$=(2x+1)^2$
$= 2x^2+1$

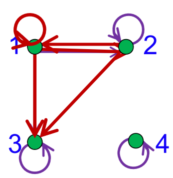


Chapter 8 - Relations

Consider the relations on {1,2,3,4}
Know which are reflexive (3 & 5)
Symmetric (2 & 3)
Transitive: (4, 5, 6)
R_1={(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)}
R_2={(1,1),(1,2),(2,1)}
R_3={(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)}
R_4={(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)}
R_5={(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,3),(4,4)}
R_6={(3,4)}

Transitivity

Consider the relations on {1,2,3,4}
- R_5={(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,3),(4,4)}



x⟶y	y⟶z	x⟶z
(2, 1)	(1, 3)	(2, 3)
(1, 2)	(2, 3)	(1, 3)

Using Matrices to represent Relations

Express the relation R_1 as a matrix
A={1.2.3.4}. B={1,2,3,4}
R1={(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)}
Think of the (a, b) pairs and remember that A=Rows and B= Cols

B

A	1	2	3	4
1	(1,1)	(1, 2)	(1,3)	(1,4)
2	(2,1)	(2, 2)	(2,3)	(2,4)
3	(3,1)	(3, 2)	(3,3)	(3,4)
4	(4,1)	(4, 2)	(4,3)	(4,4)

Put a l in each of the Corresponding locations

$$M_{R1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Is R1 Reflexive? No
Is R1 Symmetric? No
Is R1 Antisymmetric? No

Join and Meet - Matrices

For any two binary (n x n) matrices A & B
We define A ∨ B and A∧B to be the n x n binary matrices whose ij element is given by:
$(A \text{ join } B)_{ij} = A_{ij} \vee B_{ij}$ &
$(A \text{ meet } B)_{ij} = A_{ij} \wedge B_{ij}$, respectively

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}} \vee \underset{B}{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}} = \underset{A \vee B}{\begin{bmatrix} 1\vee 1 & 1\vee 0 \\ 0\vee 1 & 0\vee 0 \end{bmatrix}} = \underset{A \vee B}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}$$

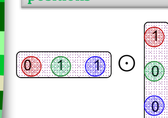
$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}} \wedge \underset{B}{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}} = \underset{A \wedge B}{\begin{bmatrix} 1\wedge 1 & 1\wedge 0 \\ 0\wedge 1 & 0\wedge 0 \end{bmatrix}} = \underset{A \wedge B}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}$$

Boolean Product Example

$(A \odot B)_{23}$ ← 3rd Col in B, 2nd Row in A

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \odot \underset{B}{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}$$

Check if there is a 1 appearing in the corresponding positions



gives 0 because
$(0 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 0)$

Some Notations - Composition

For Matrices: $M_{S \circ R} = M_R \odot M_S$

Note: The Matrix representing the composition $S \circ R = M_R \odot M_S$

Note the Ordering

$M_{R^2} = M_{R \circ R} = M_R \odot M_R = M_R^{[2]}$

$M_{R^n} = M_{R \circ R \circ \ldots \circ R} = M_R \odot M_R \odot \ldots \odot M_R = M_R^{[n]}$

n times n times

Example

- Find the Matrix Representing $M_R^{[2]}$

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_R^{[2]} = \overset{M_R}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}} \odot \overset{M_R}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}} = \overset{M_R^{[2]}}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$$

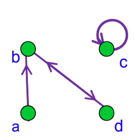
Represent Relations using Digraphs

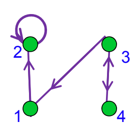
Digraph means "directed graph"

- Means there is an arrow on the arcs connecting the vertices indicating direction

For example:

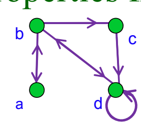
R={(a,b), (b,d), (c,c), (d,b)} R={(1,2), (2,2), (3,1),(3,4) (4,3)}





- Every diagraph represents a relation R on a set A

Finding properties from Digraphs



Reflexive? No - missing loops at a, b, c

Irreflexive? No - it has a loop at d

Symmetric? No - single directions at (b,c) and (c,d)

Antisymmetric? No - edges (a,b) and (b,d) go both directions

Asymmectric? Neither Irreflexive or Antisymmectric

Transitive? No – there is an edge (a,b) and (b,c) but no (a,c)

Finding Closures from a Digraph

- To find the reflexive closure
  - add loops.
- To find the symmetric closure
  - add arcs in the opposite direction.
- To find the transitive closure - if there is a path from a to b
  - add a direct arc from a to b.

Note: Reflexive and Symmetric closures are easy
Transitive can be complicated

And with a Matrix

- To find the reflexive closure
  - Put 1's on the diagonal.
- To find the symmetric closure
  - Take the transpose M^T of the connection matrix M_R

Transitive Closures: Diagraphs

Constructing $R^* = R \cup R^2 \cup \ldots \cup R^n$ using Digraphs:

Given R draw the corresponding diagraph D

Then compute

R=end points of paths of length 1 in D.

R^2=endpoints of paths of length 2 in D

.
.
.

R^n=endpoints of paths of length n in D

Then compute $R^* = R \cup R^2 \cup \ldots \cup R^n$

→ this is the transitive closure of R.

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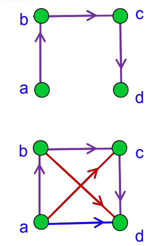
Example: Digraphs

- A={a,b,c,d} ← N=4 n so you go to R^4
- R={(a,b),(b,c),(c,d)}
- The digraph of R is:



We get R^2={ (a, c), (b, d) }

R^3={ (a, d) }

R^4=∅

Then

$R^* = R \cup R^2 \cup R^3 \cup R^4$

={(a,b),(b,c),(c,d),(a,c),(b,d),(a,d)}

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Trans. Closure: Binary Matrices

Constructing $R^* = R \cup R^2 \cup \ldots \cup R^n$ using Binary Matrices

Given A and the relation R on A, construct the matrix M_R associated to R.

Then build the powers

$R \leftrightarrow M_R^{[2]} = M_R \odot M_R$

$R \leftrightarrow M_R^{[3]} = M_R \odot M_R \odot M_R$

.
.

$R \leftrightarrow M_R^{[n]} = M_R \odot \ldots \odot M_R$ ← N Times

The matrix associated to $R^* = R \cup R^2 \cup \ldots \cup R^n$ is

$M_{R^*} = M_R \vee M_R^{[2]} \vee \ldots \vee M_R^{[n]}$

Once we get M_{R^*} it is very easy to write down R^*

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Example: Matrices

- A={a,b,c,d}
- R={(a,b),(b,c),(c,d)}

$$M_R = \begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{bmatrix}$$

$$M_R^{[2]} = \begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{bmatrix} \odot \begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{bmatrix} = \begin{bmatrix} 0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix}$$

$$M_R^{[3]} = M_R \odot M_R^{[2]} = \begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{bmatrix} \odot \begin{bmatrix} 0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix} = \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$$

Note the order

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Example: Matrices (2)

$$M_R^{[4]} = M_R \odot M_R^{[3]} = \begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{bmatrix} \odot \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$$

So we get:

$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee M_R^{[4]}$

which gives

R^* ={(a,b),(b,c),(c,d),(a,c),(b,d),(a,d)}

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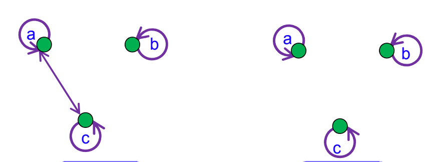
Equivalence Relations

RST → Reflexive, Symmetric, Transitive

EQUIVALENCE CLASSES:

It is easy to recognize equivalence Classes using digraphs.



Rank=2

[a]={a,c}, [c]={a.c}. [b]={b}

Rank=3

[a]={a}, [b]={b}. [c]={c}

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Partitions

- The equivalence classes of an equivalence relation R *partition* the set A into disjoint, nonempty subsets whose union is the entire set

Let S be a set = {1,2,3,4,5}

Which are partitions of S?

T_1={1,2}, T_2={3}, T_3={4,5} Yes

T_1={1,2,3}, T_2={2,4}, T_3={5} No

T_1={1}, T_2={2,3}, T_4={4} No

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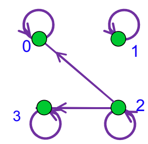
POSETs

- Partial order that is
- RAT -> Reflexive, Antisymmetric, Transitive
- If two sets are always related

 it is a total order



The elements 1,3 are incomparable

(because $(1, 3) \notin R$ and $(3, 1) \notin R$) so (A,R) is not a total ordered set.

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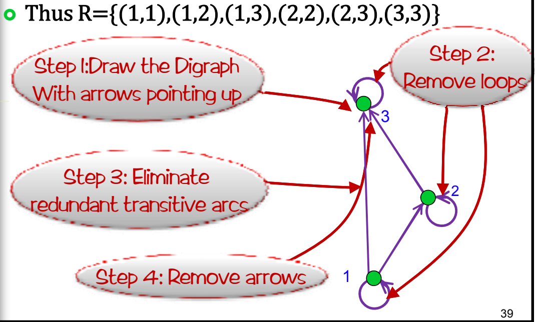
Hasse Diagram

Construct the Hasse diagram of

A={1, 2, 3}, R = ≤

- Thus R={(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)}



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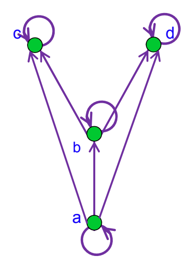
Hasse Diagram to a Digraph-- Find Ordered Pairs



- Okay, so what do we know about this diagram
- We know it is antisymmetric
  - and that the arrows point up
- We know it is reflexive
- We also know it is transitive
- Now that we have the diagraph R is easy to find

R={(a,a),(a,b),(a,c),(a,d),(b,b),(b,c), (b,d),(c,c),(c,d),(d,d)}

- Know Bounds and Lattices

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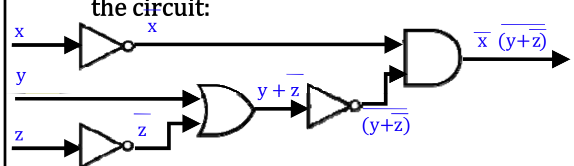
Chapter 11: Find a Boolean function from a circuit

Find the Boolean function represented by the circuit:



So this circuit represents the boolean function

$F(x,y,z) = \overline{x}\,\overline{(y+\overline{z})}$

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Find a circuit that performs a given function

Suppose you have a majority voting system with 3 individuals. Design a circuit that determines whether a proposal passes.

Let x,y,z be the votes of 3 people. The proposal passes if at least 2 people vote yes.

So, F(x,y,z)=1 if x=y=1 or x=z=1 or y=z=1.

Equivalently, F(x,y,z=1) if xy=1 or xz=1 or yz=1

Then we can represent F by

F(x,y,z)= xy+yz+xz

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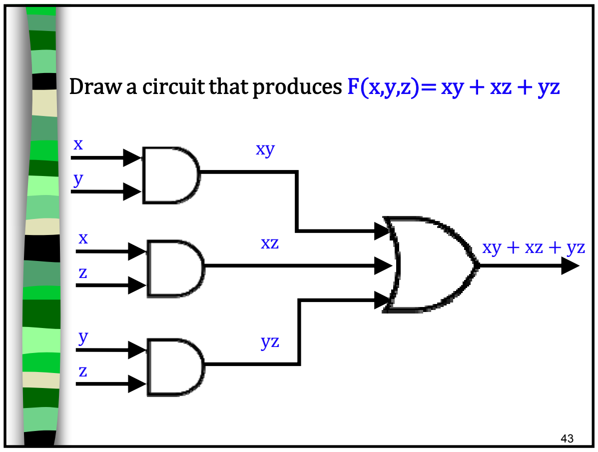
Draw a circuit that produces F(x,y,z)= xy + xz + yz
x
y
xy
x
z
xz
xy + xz + yz
y
z
yz
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